

LYAPUNOV EXPONENTS OF THE HODGE BUNDLE OVER STRATA OF QUADRATIC DIFFERENTIALS WITH LARGE NUMBER OF POLES

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ABSTRACT. We show an upper bound for the sum of positive Lyapunov exponents of any Teichmüller curve in strata of quadratic differentials with at least one zero of large multiplicity. As a corollary it stands for all Teichmüller curves in these strata and $SL(2, \mathbb{R})$ invariant subspaces defined over \mathbb{Q} . This solves Grivaux-Hubert's conjecture about the asymptotics of Lyapunov exponents for strata with large number of poles in the situation when at least one zero has large multiplicity.

1. INTRODUCTION.

Lyapunov exponents for translation surfaces have been introduced two decades ago by Zorich in [Zor99] and [Zor97] as a dynamical invariants which describe how associated leaves *wind around the surface*. On any translation surface we can introduce a translation flow which generalizes the linear flow on a flat torus (see [Zor06] for a very nice introduction to the subject). This flow has a most simple local dynamic, it is a parabolic system. Nonetheless the homology of the translation flow presents a nice asymptotic regularity and its deviation to the asymptotic cycle is described by Lyapunov exponents of the Hodge bundle over some $SL_2(\mathbb{R})$ invariant subspace in the moduli space of curves.

Even if a numerical approximation of these exponents is accessible [D⁺16], there is a priori no hope for an explicit formula to compute them. Yet a breakthrough of Eskin, Kontsevich and Zorich showed an astonishing formula binding the sum of the Lyapunov exponents to the Siegel-Veech constant of the invariant locus [EKZ14], following-up an insightful observation of [Kon97] that this sum is related to the degree of a Hodge subbundle, the formula was shown latter on in [For02]. This was the starting point to evaluate Lyapunov exponents in certain particularly symmetric cases, for example for square-tiled cyclic covers [FMZ14], [EKZ11] and triangle groups [BM10], and compute explicitly diffusion rate of wind-tree models [DHL14], [DZ15].

Other advances have been made since to estimate Lyapunov exponents for higher genus. In [Yu14], Yu gave a partial proof the conjecture of [KZ97] that the second Lyapunov exponent for hyperelliptic components of strata should go to 1 when genus goes to infinity. His proof was conditional to a conjecture which brought new ideas to find bounds from below to these exponents. His conjecture

has recently been proven in [EKMZ16], see also [DD15] for a insightful foliation point of view. Yu also obtained upper bound for the sum of Lyapunov exponents depending on Weierstrass semi-groups. Yu's idea exposed in [YZ12] and [YZ13] that independently appeared in [CM12] and [CM14] is to use algebraic characterization of sum of Lyapunov exponents and estimate it with homological algebra arguments.

In parallel Grivaux and Hubert remarked that some Teichmüller curves whose Lyapunov exponents are all zero can appear in quadratic strata. Moreover in [GH14] they prove that for this to happen the curve should be in a stratum with at least $\max(2g - 2, 2)$ poles. The heuristic explanation they provide with this result is that *poles slow down the linear flow*, since passing by a pole is like doing a U-turn for the linear flow.

Conjecture (Grivaux-Hubert). *Positive Lyapunov exponents associated to a translation surface have a uniform bound depending only on the number of poles of the surface. This bound goes to zero when the number of poles goes to infinity.*

In [CM14] Theorem 8.1, D. Chen and M. Möller obtained a result in this direction for $\mathcal{Q}(n, -1^n)$ and $\mathcal{Q}(n, 1, -1^{n+1})$, showing that these strata are non-varying and computing explicitly the sum of Lyapunov exponents for every Teichmüller curve which is equal to $2/(n + 2)$.

Using methods inspired by Yu's homological algebraic methods we obtain a general upper bound with respect to the highest multiplicity of zeros and genus. This proves the conjecture in this case.

Theorem. *For any Teichmüller curve \mathcal{C} in a quadratic stratum $\mathcal{Q}(m_1, \dots, m_k, -1^p)$ of genus g where $m_1 \geq m_2 \geq \dots \geq m_k$, if $m_1 \geq 2g$,*

$$L^+(\mathcal{C}) \leq \frac{(3g - 1)g}{m_1 + 2}$$

Where $L^+(\mathcal{C})$ stands for the sum of its positive Lyapunov exponents.

This bound can be used for more general loci in strata applying the following theorem.

Theorem ([EBW14], Theorem 5 and [EMM15], Theorem 2.3). *Let \mathcal{N}_n be a sequence of affine manifolds and suppose the sequence of affine measure $\nu_{\mathcal{N}_n} \rightarrow \nu$. Then ν is a probability measure, and it is an affine measure $\nu_{\mathcal{N}}$ where \mathcal{N} is the smallest submanifold such that there exists n_0 such that $\mathcal{N}_n \subset \mathcal{N}$ for all $n > n_0$. Moreover the Lyapunov exponents of $\nu_{\mathcal{N}_n}$ converge to the Lyapunov exponents of ν .*

As a direct consequence of the two theorems, any $\mathrm{SL}(2, \mathbb{R})$ invariant loci in a stratum with a large multiplicity zero, which can be obtained as the limit of its Teichmüller curves satisfies the bound of the former theorem and Grivaux-Hubert's conjecture. In particular for strata themselves and more generally for affine manifolds defined over \mathbb{Q} .

Corollary. *Let \mathcal{N} an affine manifold defined on \mathbb{Q} in a stratum $\mathcal{Q}(m_1, \dots, m_k, -1^p)$ of genus g where $m_1 \geq m_2 \geq \dots \geq m_k$, if $m_1 \geq 2g$,*

$$L^+(\mathcal{N}) \leq \frac{(3g-1)g}{m_1+2}$$

Where $L^+(\mathcal{C})$ stands for the sum of its positive Lyapunov exponents.

Pascal Hubert pointed out latter on that Remark 2 of [DZ15] gives a counter-example to the conjecture in general with an bounded multiplicity zeros. Taking a cover of a base surface, it exhibits an family of surfaces of genus 1 with arbitrary large number of poles, which first Lyapunov exponent is always equal to $2/3$. Thus the conjecture cannot be true in the above generality. Nonetheless broad computer experiments on strata with number of zeros and poles all going to infinity seem to show that the sum of their Lyapunov exponents goes to zero at speed $1/\sqrt{p}$. We state the following conjecture,

Conjecture. *Let Q_n be a sequence of connected components of strata for a fixed genus and p_n be their number of poles going to infinity, then*

$$\lambda_1^+(Q_n) = \mathcal{O}(1/\sqrt{p_n})$$

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2. BACKGROUND MATERIAL.

Strata. A half-translation surface is a pair (X, q) where X is a Riemann surface and a quadratic differential on X with possibly simple poles. If $S(q)$ is the set of zeros and poles of q on X , we can endow $\tilde{X} := X \setminus S(q)$ with charts $\phi_i : U_i \rightarrow X$ such that $\phi_i^* q = dz^{\otimes 2}$. In such an atlas, the transition functions are translations composed with $\pm \text{Id}$. The quadratic differential induces a flat metric $|q|$ on the open surface \tilde{X} . This metric can be extended to the whole surface and would have conical points at $S(q)$ with angles multiples of π . If we fix integers m_1, \dots, m_k, p such that $\sum_{i=1}^k m_i - p = 4g - 4$, for some positive integer g , the stratum of half-translation surfaces $\mathcal{Q}(m_1, \dots, m_k, -1^p)$ is the set of half-translation surfaces (X, q) where q has k distinct zeros of multiplicity m_1, \dots, m_k and p simple poles. The projectivized space $P\mathcal{Q}(m_1, \dots, m_k, -1^p)$ is obtained by quotient under the scalar action of \mathbb{C}^* on the quadratic differential.

Teichmüller curves. There is a natural action of $\text{GL}(2, \mathbb{R})$ on every stratum. Take the flat atlas as above, the charts were constructed in such way that transition functions are translations by vectors v_{ij} composed with $\pm \text{Id}$. For a matrix $M \in \text{GL}(2, \mathbb{R})$ we define a new surface by multiplying the previous change of charts by M . They will be translations by vectors Mv_{ij} composed with $\pm \text{Id}$.

Let $\mathbb{H} = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2, \mathbb{R})$ denote the Poincaré upper half-plane. For any (X, q) the action of $\mathrm{SL}(2, \mathbb{R})$ factors to a map

$$\mathbb{H} \rightarrow PQ(m_1, \dots, m_k, -1^p)$$

which is an immersion. The image of this map is called a Teichmüller disk. This map also factors through the Veech group which is its stabilizer. If the Veech group Γ of a given half-translation surface is a lattice in $\mathrm{SL}(2, \mathbb{R})$ we say that the surface is a Veech surface and the image of a \mathbb{H}/Γ in the projective stratum is called a Teichmüller curve. An other point of view for the Teichmüller curve is to consider the induced algebraic map $\mathcal{C} \rightarrow \mathcal{M}_g$. This is the convention used in [Möl13].

By definition a Teichmüller curve is a surface with a hyperbolic structure, eventually with cusps. A main tool for studying these curves is their compactification.

Let $f : \mathcal{X} \rightarrow C$ be a smooth family of curves of genus g over the smooth curve C , i.e. a smooth morphism with connected fibers, which are smooth curves of genus g . By definition of the moduli space, such a family induces a map $m : C \rightarrow \mathcal{M}_g$. The delicate part in the compactification of a Teichmüller curve is the choice of the surfaces we put at the cusps. Consider the extended map $\overline{m} : \overline{C} \rightarrow \overline{\mathcal{M}}_g$, it is not always true that there is a family of stable curve $\overline{f} : \overline{\mathcal{X}} \rightarrow \overline{C}$ which extends f and induces \overline{m} , but there is an unramified finite index cover $B \rightarrow C$, such that we can define a stable extension on the pullback $\overline{\mathcal{X}} \rightarrow \overline{B}$ (see [Möl13]).

The structure sheaf of $\overline{\mathcal{X}}$ will be written \mathcal{O} , and the pullback of the dualizing sheaf of $\overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_{g,n}$, will be referred to as $\omega_{\overline{\mathcal{X}}/\overline{B}}$.

Divisors classes. For a Teichmüller curve generated by a quadratic differential q , there is an associated line bundle $\mathcal{L} \subset f_* \omega_{\overline{\mathcal{X}}/\overline{B}}^{\otimes 2}$ on \overline{B} whose fiber over the point corresponding to (X, q) is $\mathbb{C} \cdot q$. This bundle has a maximality property (see [Möl13], [Gou12] and [Wri12]) which implies that $\deg(\mathcal{L}) = \chi$.

We denote by S_1, \dots, S_k and P_1, \dots, P_p sections of $\overline{\mathcal{X}} \rightarrow \overline{B}$, which intersect the fibers at each zero with multiplicity m_1, \dots, m_k and at poles. For convenience we also introduce the total divisor $\mathcal{A} := \sum_{i=1}^n m_i Z_i - \sum_{j=1}^p P_j$ and the divisor of zeros $\mathcal{Z} := \sum_{i=1}^n m_i Z_i$. We have the following exact sequence induced by the pullback of the inclusion of \mathcal{L} ,

$$0 \rightarrow f^* \mathcal{L} \rightarrow \omega_{\overline{\mathcal{X}}/\overline{B}}^{\otimes 2} \rightarrow \mathcal{O}(\mathcal{A}) \rightarrow 0$$

This is true for non compactified Teichmüller spaces, and it stays true after compactification since the multiplicities of the zeros of the limit differentials stay the same on stable curves.

According to the adjunction formula,

$$(\omega_{\overline{\mathcal{X}}/\overline{B}} + S_i)|_{S_i} = \omega_{S_i} = 0$$

Which implies $S_i^2 = -\omega_{\overline{\mathcal{X}}/\overline{B}} \cdot S_i$. Moreover $S_i \cdot S_j = 0$ for any $i \neq j$, since two fibers of f have zero intersection. Finally,

$$2S_i^2 = -m_i S_i^2 - \deg \mathcal{L}$$

Which leads to the formula :

$$(1) \quad S_i^2 = \frac{-\chi}{m_i + 2}$$

3. LYAPUNOV EXPONENTS

Lyapunov exponents are dynamical invariants for translation surfaces which describe the behaviour of the translation flow on them. For an introduction to their multiple aspects in translation surfaces see [Zor06]. We will adopt an algebraic point of view to compute them. It is based on the observation of [Kon97] that their sum is related to the degree of a Hodge subbundle on a Teichmüller curve (see [EKZ14] or [For02] for a proof). We will use in the following a more precise formula computed in [CM14] as (20),

$$L^+(\mathcal{C}) = \frac{2}{\chi} \cdot \deg f_* \mathcal{O}(\mathcal{A}) + (6g - 6) - \frac{1}{2} \left(\sum_{i=1}^n \frac{m_i(m_i + 4)}{m_i + 2} - 3p \right)$$

Where $L^+(\mathcal{C})$ is the sum of the positive Lyapunov exponents associated to the Teichmüller curve \mathcal{C}

By Gauss-Bonnet,

$$\begin{aligned} \sum_{i=1}^n m_i - p &= 4g - 4 \\ 6g - 6 + \frac{3}{2}p &= \frac{3}{2} \sum_{i=1}^n m_i \end{aligned}$$

Remark that $m_i(m_i + 4) - 3m_i(m_i + 2) = -2m_i(m_i + 1)$, this leads to the following formula with a form which will be effective for us here :

$$(2) \quad L^+(\mathcal{C}) = \frac{2}{\chi} \cdot \deg f_* \mathcal{O}(\mathcal{A}) + \sum_{i=1}^n \frac{m_i(m_i + 1)}{m_i + 2}$$

This equation relates the sum of Lyapunov exponents for a Teichmüller curve to a term defined as an invariant of the stratum and a term describing the behaviour of the curve at its cusps given by the Chern class of a sheaf.

To estimate the sum of Lyapunov exponents, we are reduced to bound the degree of the sheaf $f_* \mathcal{O}(\mathcal{A})$. This will be done based on the two following lemmas which proof will be given in the next sections.

Lemma 1.

$$\deg f_* \mathcal{O}(\mathcal{A}) = \deg f_* \mathcal{O}(\mathcal{Z})$$

Thus the pole divisors won't interfere in the degree we need to estimate. By a similar method, we will be able to bound the degree while taking out zeros one after the other. We will do so until no zero is left and end up with the canonical sheaf. But first we need to choose an order in which we will remove the zeros.

Let us introduce some notations first. We denote by \underline{d} a collection of non negative integers (d_1, \dots, d_k) , $|\underline{d}| := d_1 + \dots + d_k$ and $\underline{d}\mathcal{Z} := \sum_{i=1}^k d_i Z_i$.

We pick a finite family $\mathcal{F} := \{\underline{d}\}^{i \geq 1}$ satisfying the properties

- (1) The first term is $\underline{d}^1 = (m_1, \dots, m_k)$
- (2) The last term is $\underline{0}$
- (3) If $\underline{d}^i = (d_1, \dots, d_k)$ and $\underline{d}^{i+1} = (d'_1, \dots, d'_k)$, for some n , $d_n = d'_n + 1$ and $d_m = d'_m$ for all other indices.

We use in the following the notations $\delta^i := d_n$, $\mu^i := m_n$, $\Sigma^i := Z_n$ and

$$\mathcal{I}^i := \frac{2}{\chi} \cdot \delta^i Z_n \cdot \underline{d}^i \mathcal{Z} = \frac{2\delta^i}{\mu^i + 2}$$

These properties imply that the indices should vary between $1 \leq i \leq 4g - 4 + p$.

We associate to such a family its gaps $G(\mathcal{F}) = i_1, \dots, i_g$, *i.e.* indices l defined by the property that on a generic fiber F ,

$$h^0(\underline{d}^l \mathcal{Z}|_F) = h^0(\underline{d}^{l+1} \mathcal{Z}|_F)$$

According to Weierstrass group theory (see [YZ13] and [ACG11] for an introduction), there will be just g gaps.

Lemma 2. *If $i \in G(\mathcal{F})^c$,*

$$\deg f_* \mathcal{O}(\underline{d}^i \mathcal{Z}) \leq \deg f_* \mathcal{O}(\underline{d}^{i+1} \mathcal{Z}) - \delta^i \frac{\chi}{\mu^i + 2}$$

If $i \in G(\mathcal{F})$,

$$\deg f_* \mathcal{O}(\underline{d}^i \mathcal{Z}) = \deg f_* \mathcal{O}(\underline{d}^{i+1} \mathcal{Z})$$

By recurrence, we obtain the inequality,

$$\deg f_* \mathcal{O}(\mathcal{A}) \leq \sum_{i=1}^k \frac{m_i(m_i + 1)}{2} \cdot \frac{-\chi}{m_i + 2} + \sum_{l \in G(\mathcal{F})} \mathcal{I}^l$$

Using Lemma 1 and formula 2,

$$(3) \quad L^+(\mathcal{C}) \leq \sum_{l \in G(\mathcal{F})} \mathcal{I}^l$$

This will lead to a proof of the Theorem written down in the introduction. Assume we are in the setting of this theorem, we denote the larger zero multiplicity by $m_1 \geq 2g - 1$. We pick a family \mathcal{F} such that we first remove all zeros different from the first one in an arbitrary way, and finish by taking off all the multiplicity of the principal one. Remark that for $l \in G(\mathcal{F})$,

- $\delta^l \leq |\underline{d}^l|$
- $|\underline{d}^l|$ is decreasing w.r.t. l and is less than $2g - 1$
- $\mu^l = m_1$ for every gap

where the two last properties are implied by Riemann-Roch theorem.

Thus,

$$\sum_{l \in G(\mathcal{F})} \mathcal{I}^l \leq \sum_{i=1}^g \frac{2(2g-i)}{m_1+2} \leq \frac{(3g-1)g}{m_1+2}$$

Remark. *If we can show that one of the zeros with multiplicity larger than g is not a Weierstrass point generically, the bound becomes $\frac{(g+1)g}{m_1+2}$.*

4. PROOF OF LEMMAS

Lemma 1. The main tool in the introduction of this filtration is the structural exact sequence for any divisor D in $\overline{\mathcal{X}}$

$$0 \rightarrow \mathcal{O}(-D) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_D \rightarrow 0$$

Assume here that D is one of the divisors P_j . We tensorize this exact sequence by $f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A} + D)$, where \mathcal{A} is the total divisor as introduced in section 2. The functor f_* implies a long exact sequence,

$$\begin{array}{ccccccc} 0 & \rightarrow & f_*(\omega^{\otimes 2}) & \rightarrow & \mathcal{L} \otimes f_*\mathcal{O}(\mathcal{A} + D) & \rightarrow & \mathcal{L} \\ & & \xrightarrow{\delta} & & R^1 f_*(\omega^{\otimes 2}) & \rightarrow & R^1 f_*(f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A} + D)) \rightarrow 0 \end{array}$$

Where we use the fact that f induces an isomorphism between D and $\overline{\mathcal{X}}$, which implies the equalities $f_*\mathcal{O}_D = \mathcal{O}_{\overline{\mathcal{X}}}$ and $\omega^{\otimes 2} = f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A})$.

Remark that $R^1 f_*(\omega^{\otimes 2})$ over each point X of the Teichmüller curve has dimension

$$h^1(X, \omega_X^{\otimes 2}) = h^0(X, \omega_X^{\otimes -1}) = 0$$

by Serre's duality. Hence this sheaf is zero, and so is $R^1 f_*(f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A} + D))$. We end up with a short exact sequence which implies a relation on the degrees,

$$\deg(\mathcal{L} \otimes f_*\mathcal{O}(\mathcal{A} + D)) = \deg(\mathcal{L} \otimes f_*\mathcal{O}(\mathcal{A})) + \deg(\mathcal{L})$$

We start again with $f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A} + D)$, as we noticed its image through the derived functor $R^1 f_*$ is zero and so will be the image of $f^*\mathcal{L} \otimes \mathcal{O}(\mathcal{A} + D + D')$

for any other D' divisor picked in the P_j . We do this until all the pole divisors are consumed, and get a formula on degrees implying the divisor without poles \mathcal{Z} ,

$$\deg(\mathcal{L} \otimes f_* \mathcal{O}(\mathcal{Z})) = \deg(\mathcal{L} \otimes f_* \mathcal{O}(\mathcal{A})) + p \cdot \deg(\mathcal{L})$$

To finish, remark that fibers of $f_* \mathcal{O}(\mathcal{Z})$ have constant dimension $3g - 3 + p$ by Riemann-Roch theorem, since the degree of the divisor on any fiber will be $4g - 4 + p > 2g - 2$ for $g, p \geq 1$. By Grauert Semicontinuity Theorem (see for example [Har77]) this sheaf will be a vector bundle. And so will be $f_* \mathcal{O}(\mathcal{A})$ as it is a subsheaf of the later. Moreover $\text{rk } f_* \mathcal{O}(\mathcal{Z}) = 3g - 3 + p$ and $\text{rk } f_* \mathcal{O}(\mathcal{Z}) = 3g - 3$.

The previous formula becomes

$$(3g - 3 + p) \deg(\mathcal{L}) + \deg(f_* \mathcal{O}(\mathcal{Z})) = (3g - 3) \deg(\mathcal{L}) + \deg(f_* \mathcal{O}(\mathcal{A})) + p \cdot \deg(\mathcal{L})$$

And finally,

$$\deg(f_* \mathcal{O}(\mathcal{Z})) = \deg(f_* \mathcal{O}(\mathcal{A}))$$

Lemma 2. Similarly to the previous subsection we tensor the structural exact sequence for the divisor Σ^i by $f_* \mathcal{O}(\underline{d}^{i+1} \mathcal{Z})$, we get the following long exact sequence,

$$0 \rightarrow f_* \mathcal{O}(\underline{d}^{i+1} \mathcal{Z}) \rightarrow f_* \mathcal{O}(\underline{d}^i \mathcal{Z}) \rightarrow \mathcal{O}_{\Sigma^i}(\delta^i \Sigma^i) \xrightarrow{\delta} \dots$$

Which induces the formula,

$$\deg f_* \mathcal{O}(\underline{d}^i \mathcal{Z}) - \deg f_* \mathcal{O}(\underline{d}^{i+1} \mathcal{Z}) = \deg \ker \delta$$

As $\ker \delta$ is a subsheaf of $\mathcal{O}_{\Sigma^i}(\delta^i \Sigma^i)$ which is locally free on a complex curve, both are locally free and when $\ker \delta$ is not zero, *i.e.* when $i \in G(\mathcal{F})^c$

$$\deg \ker \delta \leq \deg \mathcal{O}_{\Sigma^i}(\delta^i \Sigma^i) = \delta^i \Sigma^i \cdot \Sigma^i$$

Hence by 1,

$$\deg f_* \mathcal{O}(\underline{d}^i \mathcal{Z}) \leq \deg f_* \mathcal{O}(\underline{d}^{i+1} \mathcal{Z}) - \delta^i \frac{\chi}{\mu^i + 2}$$

If $i \in G(\mathcal{F})$, $\ker \delta = 0$ thus

$$f_* \mathcal{O}(\underline{d}^i \mathcal{Z}) \simeq f_* \mathcal{O}(\underline{d}^{i+1} \mathcal{Z})$$

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